

k-core percolation on multiplex networks

N. Azimi-Tafreshi,¹ J. Gómez-Gardeñes,^{2,3} and S. N. Dorogovtsev^{4,5}

¹*Physics Department, Institute for Advanced Studies in Basic Sciences, 45195-1159 Zanjan, Iran*

²*Departamento de Física de la Materia Condensada,
Universidad de Zaragoza, E-50009 Zaragoza, Spain*

³*Instituto de Biocomputación y Física de los Sistemas Complejos (BIFI),
Universidad de Zaragoza, E-50018 Zaragoza, Spain*

⁴*Departamento de Física da Universidade de Aveiro & I3N,
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal*

⁵*A.F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*

We generalize the theory of k -core percolation on complex networks to \mathbf{k} -core percolation on multiplex networks, where $\mathbf{k} \equiv (k_a, k_b, \dots)$. Multiplex networks can be defined as networks with a set of vertices but different types of edges, a, b, \dots , representing different types of interactions. For such networks, the \mathbf{k} -core is defined as the largest subgraph in which each vertex has at least k_i edges of each type, $i = a, b, \dots$. We derive self-consistency equations to obtain the birth points of the \mathbf{k} -cores and their relative sizes for uncorrelated multiplex networks with an arbitrary degree distribution. To clarify our general results, we consider in detail multiplex networks with edges of two types, a and b , and solve the equations in the particular case of Erdős-Rényi and scale-free multiplex networks. We find hybrid phase transitions at the emergence points of \mathbf{k} -cores except the $(1, 1)$ -core for which the transition is continuous. We apply the \mathbf{k} -core decomposition algorithm to air-transportation multiplex networks, composed of two layers, and obtain the size of (k_a, k_b) -cores.

PACS numbers: 64.60.aq, 89.75.Fb, 05.70.Fh, 64.60.ah

I. INTRODUCTION

In the last decade, during the advent of network science, a number of statistical descriptions were proposed to characterize the structure of the interactions of many diverse complex systems [1–4]. One of the most fundamental features when characterizing the structure of a network is the size of its giant connected component, *i.e.*, the size of the largest subset of vertices so that each pair of them is connected by a path of finite length. The existence of a giant connected component is particularly important in those networks carrying flows of different natures, such as viruses and rumours in social systems, data in technological networks or goods and humans in transportation systems. Thus, the maximum capacity for the spread and transport in such systems is limited by the size of the giant connected component. Apart from this practical meaning, the giant connected component is the relevant order parameter that shows the formation of a macroscopic cluster in the context of ordinary percolation [5–7].

Apart from ordinary percolation, which leads to a continuous phase transition for the emergence of a macroscopic giant connected component, a number of generalizations to the usual scenario were introduced. These variants lead to other kinds of giant connected components, whose emergence is associated with different phase transitions [8–13]. Among these generalizations is k -core percolation, in which a giant k -core exists if the vertex mean degree of the network exceeds some threshold [8, 9]. The k -core of a complex network is defined as the largest subgraph whose vertices have degree at least k . Thus, each two vertices in the k -core are interconnected by

at least k paths, which may partially overlap with each other.

The k -core of a given graph can be obtained by a recursive pruning algorithm that, at each step, removes all existing vertices with degrees less than k . As the result of this pruning, the network is decomposed into a hierarchical set of progressively enclosed k -cores with the highest k -core being placed in the center. The application of this graph decomposition technique to large real-world networks allows to describe qualitatively their structure in terms of the complete set of their k -cores [14, 15]. Moreover, it was shown that the most efficient vertices in spreading processes are those belonging to higher k -cores of a network [16]. For these reasons, during the last years the k -core organization of complex networks has been extensively studied [8, 9, 17]. The most remarkable result is that for $k \geq 3$, a k -core emerges discontinuously at the percolation threshold through a hybrid phase transition, combining a discontinuity and a critical singularity and thus breaking the usual continuous scenario of ordinary percolation.

Very recently, it has been considered that most of real-world networks are not isolated objects but composed of many coupled, interdependent networks such that the function of one network depends on the others [18, 19]. For such networks, the pruning of vertices in one network can lead to removal of dependent vertices in other networks. It was found that generalized percolation properties of these interdependent networks differ strongly from ordinary percolation on a single network [20–23]. In fact, they more resemble features of k -core percolation in single networks including the hybrid phase transition.

The simplest particular case of interdependent net-

works are multiplex networks in which each vertex depends on at most one vertex of other networks. Multiplex networks can be treated as superpositions of several networks (sometimes called layers) with different edges [24]. In other words, all vertices in these networks are of the same type, but the edges are of different types (colors). Note that here a vertex not necessary has all types of connections. Multiplexes have recently attracted a lot of attention as they are the kind of substrates representing better the interaction patterns occurring in many real systems, in which several ways for the interaction between the elements coexist. This is the case of transportation, social and technological networks among others.

Based on its ubiquity, the structural and dynamical properties of multiplexes have been studied in several contexts including diffusion [25, 26], evolutionary games [27], Boolean dynamics [28], epidemics [29] and, of course, percolation [30, 31]. In this way, in [31], a giant viable component for multiplex networks is defined as a set of vertices in which, for every type of edges, each two vertices are interconnected by at least one path following only edges of this type. Clearly, the giant viable component of a multiplex network is a subgraph of the giant connected components of all single networks (layers). The theory of ordinary percolation on multiplex networks shows a hybrid transition which is the birth of the giant viable components, similar to what happens in k -core percolation on single networks. Furthermore, in [20] it was shown that the giant viable component is a subgraph of the so-called mutual component (or mutually connected component). By definition, a vertex belongs to the mutual component if for every color of edges, which he has, at least one its nearest neighbor connected by an edge of this color is in the mutual component. Since vertices may have among their edges, some of colors missed, some pairs of vertices in the mutual component may be interconnected by paths of incomplete set of the colors.

In this paper we study the organization of specific subgraphs, \mathbf{k} -cores for multiplex networks, where $\mathbf{k} \equiv (k_a, k_b, \dots)$. Generalizing the notion of a viable connected component, we define the \mathbf{k} -viable component as a set of vertices, in which for each type i of edges and for each two vertices, there are at least k_i paths, following only edges of type i . The \mathbf{k} -core is a giant \mathbf{k} -viable component, where we must assume for each type i of edges that $k_i \geq 2$.

The paper is organized as follows. In section II, we introduce an algorithm for the \mathbf{k} -core decomposition of multiplex networks and present an analytical framework enabling us to describe the nature of the transitions corresponding to the emergence of \mathbf{k} -cores with arbitrary (k_a, k_b, \dots) . We apply our general results to the Erdős-Rényi and scale-free multiplex networks. In section III, as an application to the real multiplexes, we apply the \mathbf{k} -core algorithm to air-transportation multiplexes and compare the results with our analytical predictions.

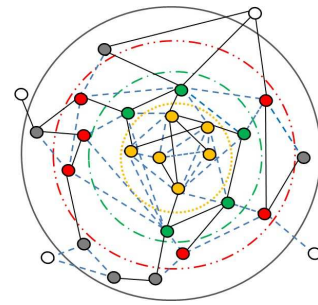


FIG. 1. The (k_a, k_b) -core decomposition for a multiplex network with two types of edges. The cores from out to center are the $(1, 1)$ -core, the $(1, 2)$ -core, the $(2, 2)$ -core, and $(1, 3)$ -core.

II. K-CORE ON MULTIPLEX NETWORKS

A. Analytical framework

Let us consider a multiplex network, having edges of types a, b, \dots , with a given joint degree distribution $P(q_a, q_b, \dots)$ and a locally tree-like structure in the infinite network limit. For simplicity, we assume that this network is completely uncorrelated, though, in principle, correlations between different edges, $i = a, b, \dots$, of a vertex might be easily taken into account. To obtain the \mathbf{k} -core of a multiplex network, we use the following pruning algorithm: at each step we remove every vertex if for at least one type of edge i , $q_i < k_i$. As the result of the pruning, the degrees of some vertices change. If there are still vertices which one can prune, we remove them in the next step. The pruning is continued until no vertex remains of degree q_i less than the threshold k_i . Fig. 1 shows the \mathbf{k} -core decomposition for a multiplex network with two types of edges.

To find the size of the giant \mathbf{k} -cores, for each type i of edges, we define x_i as the probability that an end vertex of a randomly chosen edge of type i is the root of an infinite subtree of type i . The subtree of type i is, by definition, a tree whose vertices have at least $k_i - 1$ edges of type i and at least k_j edges of each of other types $j \neq i$ edges. Probabilities x_a and x_b for a multiplex network with two types a and b of edges are schematically shown in Fig. 2. These probabilities play the role of the order parameters of the phase transition associated with the emergence of the \mathbf{k} -cores. We can write the self-consistency equations for probabilities x_i using the locally tree structure of the networks,

$$x_i = \sum_{\mathbf{q}} \frac{q_i P(\mathbf{q})}{\langle q_i \rangle} \left[\sum_{s=k_i-1}^{q_i-1} \binom{q_i-1}{s} x_i^s (1-x_i)^{q_i-1-s} \right] \times \prod_{j \neq i} \left[\sum_{s'=k_j}^{q_j} \binom{q_j}{s'} x_j^{s'} (1-x_j)^{q_j-s'} \right], \quad (1)$$

where $\mathbf{q} \equiv (q_a, q_b, \dots)$ is the degree of a vertex.

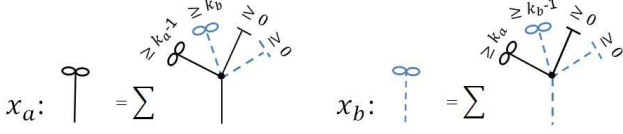


FIG. 2. Schematic representation of the self-consistency equations for the probabilities x_a and x_b . The solid black and dashed blue lines with infinity symbols at one of their ends, represent probabilities x_a and x_b respectively. The edges lead to finite components, namely $1 - x_a$ and $1 - x_b$, are shown by solid and dashed lines with cuts at one of their ends.

Let us briefly explain Eq. (1). The probability that the end vertex of a randomly chosen edge of type i has degree \mathbf{q} , is $q_i P(\mathbf{q}) / \langle q_i \rangle$. The combinatorial multiplier $\binom{m}{n}$ gives the number of ways one can choose n edges from a sample of m edges. At least $k_i - 1$ edges of $q_i - 1$ edges of type i (other edges than the starting one) must lead to an infinite subtree of type i (probability x_i) and at least k_j edges of each of q_j edges ($j \neq i$) must lead to the infinite subtrees of type j (probability x_j).

Using these probabilities, we can obtain the relative size of the giant \mathbf{k} -core. A vertex is in the \mathbf{k} -core when, for each type of edges i , the vertex has at least k_i edges leading to infinite i subtrees. The probability that a vertex belongs to the (k_a, k_b) -core for multiplex networks with edges of two types, a and b , is shown schematically in Fig. 3. Hence, the relative size of the core is given by the following expression:

$$n_{\mathbf{k}\text{-core}} = \sum_{\mathbf{q}} P(\mathbf{q}) \prod_i \left[\sum_{s=k_i}^{q_i} \binom{q_i}{s} x_i^s (1 - x_i)^{q_i - s} \right]. \quad (2)$$

We can now rewrite Eqs. (1)–(2) using generating functions [7], which enable us to solve these equations analytically. For a single network with a given degree distribution $P(q)$, the generating function $G(x)$ is defined as

$$G(x) \equiv \sum_q P(q) x^q. \quad (3)$$

If we assume that there is no correlation between the degrees q_a, q_b, \dots , so that $P(\mathbf{q}) = P(q_a)P(q_b)\dots$, then

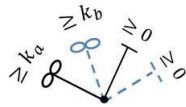


FIG. 3. Schematic representation of a vertex belongs to (k_a, k_b) -core.

Eqs. (1) and (2) can be rewritten as following

$$x_i = \left[1 - \frac{1}{\langle q_i \rangle} \sum_{s=0}^{k_i-2} x_i^s \frac{G_i^{(s+1)}(1 - x_i)}{s!} \right] \times \prod_{j \neq i} \left[1 - \sum_{s'=0}^{k_j-1} x_j^{s'} \frac{G_j^{(s')}(1 - x_j)}{s'!} \right] \quad (4)$$

and

$$n_{\mathbf{k}\text{-core}} = \prod_i \left[1 - \sum_{s=0}^{k_i-1} x_i^s \frac{G_i^{(s)}(1 - x_i)}{s!} \right], \quad (5)$$

where we used the notation $G^s(x)$ for the s -th derivatives of $G(x)$, which produce the higher moments of the distribution $P(q)$.

In general, Eq. (4) is a set of self-consistency equations of the following form $x_i = f_i(x_a, x_b, \dots)$. A hybrid transition occurs at the point in which f_i first meets x_i . This point is determined by the condition $\det[\mathbf{J} - \mathbf{I}] = 0$ for the Jacobian matrix \mathbf{J} , defined as $J_{ij} = \partial f_j / \partial x_i$, and \mathbf{I} is the identity matrix. Expanding f_i around the transition point, we find the singularity of x_i and hence $n_{\mathbf{k}\text{-core}}$, which is the relative size of the \mathbf{k} -core. If we choose $\langle q_i \rangle$ as control parameter, then the singularity at the critical point is

$$n_{\mathbf{k}\text{-core}} - n_{\mathbf{k}\text{-core}}^c \propto (x_i - x_i^c) \propto (\langle q_i \rangle - \langle q_i \rangle^c)^{1/2}. \quad (6)$$

The structure of the \mathbf{k} -core is described by the degree distribution $P_{\mathbf{k}}(\mathbf{q})$ defined as the probability to find a vertex of degree $\mathbf{q} \equiv (q_a, q_b, \dots)$ in the \mathbf{k} -core:

$$P_{\mathbf{k}}(\mathbf{q}) = \frac{n_{\mathbf{k}}(\mathbf{q})}{n_{\mathbf{k}\text{-core}}}, \quad (7)$$

where $n_{\mathbf{k}}(\mathbf{q})$ is the fraction of vertices with degree \mathbf{q} , which fall into the \mathbf{k} -core. This fraction is given by the following expression:

$$n_{\mathbf{k}}(\mathbf{q}) = \sum_{\mathbf{q}' \geq \mathbf{q}} P(\mathbf{q}') \prod_i \left[\binom{q'_i}{q_i} x_i^{q_i} (1 - x_i)^{q'_i - q_i} \right]. \quad (8)$$

In particular, Eq. (8) helps us to find the size of the corona (a subset of vertices of degree \mathbf{k} in the \mathbf{k} -core, which generalizes the notion of the corona clusters of the k -core). Setting $\mathbf{q} = \mathbf{k}$ and making use of generating functions, we find the relative size of the corona:

$$n_{\mathbf{k}}(\mathbf{k}) = \prod_i x_i^{k_i} \frac{G_i^{(k_i)}(1 - x_i)}{k_i!}. \quad (9)$$

Furthermore, the average total degree $q_a + q_b + \dots$ of a vertex in the \mathbf{k} -core can be obtained using the degree distribution of the \mathbf{k} -core in the following way:

$$c_{\mathbf{k}\text{-core}} = \sum_{\mathbf{q}} (q_a + q_b + \dots) P_{\mathbf{k}}(\mathbf{q}). \quad (10)$$

To clarify our results, in the following we will consider Erdős–Rényi and scale-free multiplex networks with two types, a and b , of edges to describe the \mathbf{k} -core organization of these multiplex networks.

B. Erdős–Rényi networks

Let us first consider Erdős–Rényi multiplex networks with Poisson degree distributions: $P(q_a) = c_a^{q_a} e^{-c_a q_a} / q_a!$, $P(q_b) = c_b^{q_b} e^{-c_b q_b} / q_b!$, where c_a and c_b are the mean vertex degrees for types a and b of edges, respectively. For the Poisson distribution, the generating function and its s -th derivative are $G_i(x) = e^{-c_i(1-x)}$ and $G_i^s(x) = c_i^s e^{-c_i(1-x)}$, respectively.

For the sake of simplicity, let us consider the symmetric case $c_a = c_b \equiv c$. The largest core is the core with $k_a = k_b = 1$, that is $(1,1)$ -core. In this case one can obtain $x_a = x_b \equiv X$, such that $X = 1 - e^{-cX}$. For $c > 1$, this equation has a nonzero nontrivial solution. Figure 4(a) shows the relative size of the $(1,1)$ -core, displaying a continuous transition at $c = 1$ and is given by the following expression

$$n_{(1,1)\text{-core}} = (1 - e^{-cX})^2 \cong 4(c-1)^2, \quad (11)$$

Notice that the case of the $(1,1)$ -core is special since it does not coincide with the viable component [23, 31]. Between each two vertices in the $(1,1)$ -core, there are at least 2 distinct paths within the core, following edges that can alternately change from one type to the another. Recall that, in contrast to this, between each two vertices in the viable component, for each type of edges, there is at least one path following only edges of that type. The fact that the $(1,1)$ -core does not coincide with the viable component is not surprising. Indeed, in ordinary single networks, the 1-core also does not coincide with the giant connected component (*i.e.* the percolation cluster). Instead, it is rather the 2-core that is close to the giant connected component.

Furthermore, if only one of the two k_i values is equal to 1, the (k_a, k_b) -core is still not a (k_a, k_b) -viable component in the sense that there are still not $k_a + k_b$ paths between two vertices following edges of the corresponding types. However, in this case there are $k_a + k_b \geq 3$ paths between each two vertices within the (k_a, k_b) -core following edges of alternating types. For instance, let us consider the $(1,2)$ -core. In this case, one can find the following equations for x_a and x_b :

$$\begin{aligned} x_a &= (1 - e^{-c_b x_b} - c_b x_b e^{-c_b x_b}), \\ x_b &= (1 - e^{-c_b x_b})(1 - e^{-c_a x_a}). \end{aligned} \quad (12)$$

Using these probabilities, the relative size of the $(1,2)$ -core is given by the following expression:

$$n_{(1,2)\text{-core}} = (1 - e^{-c_a x_a})(1 - e^{-c_b x_b} - c_b x_b e^{-c_b x_b}). \quad (13)$$

Fig. 4(a), in particular, shows the relative size of the $n_{(1,2)\text{-core}}$ in the symmetric case of $c_a = c_b = c$. While the giant core is non-viable, a hybrid transition appears at the emergence point of the core at $c \simeq 2.7461$.

Let us now assume that each of k_i exceeds 1. In this case, considering the tree for x_i in Fig. 2, one can show that this \mathbf{k} -core is a \mathbf{k} -viable component in the sense

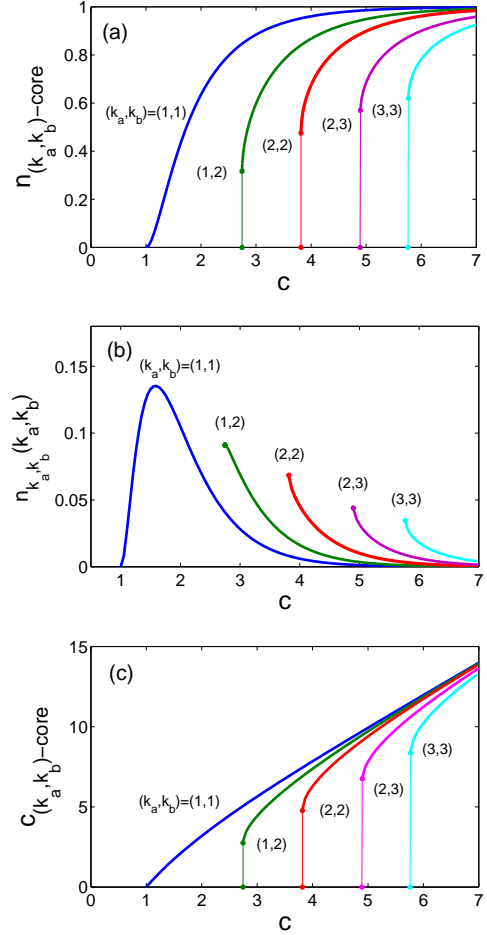


FIG. 4. The relative sizes of (a) \mathbf{k} -core, (b) corona and (c) mean vertex degree of \mathbf{k} -core in Erdős–Rényi multiplex networks for some values of k_a and k_b , vs the vertex mean degree $c_a = c_b = c$ of the network.

that for every type i of edges, each two vertices within the core are interconnected by at least k_i paths following only edges of this type (see Fig. 3). As an example, let us consider $k_a = k_b = 2$. In the symmetric case of $c_a = c_b \equiv c$, we find $x \equiv x_a = x_b$ satisfying the following equation:

$$x = (1 - e^{-cx})(1 - e^{-cx} - cxe^{-cx}). \quad (14)$$

Solving the equation for x , we can find that the transition occurs at $c \simeq 3.8166$. The relative size of $(2,2)$ -core is given by the expression:

$$n_{(2,2)\text{-core}} = (1 - e^{-cx} - cxe^{-cx})^2. \quad (15)$$

This core emerges discontinuously at the transition point, as shown in Fig. 4(a). Fig. 4(a) also shows the relative sizes of the (k_a, k_b) -cores for some other values of k_a and k_b . There is a jump at the birth points of the cores, which, together with a critical singularity, points out a hybrid transition for $k_a \geq 2$ and $k_b \geq 2$.

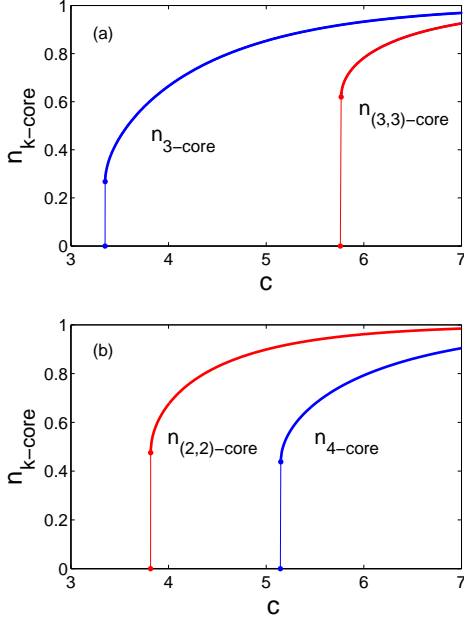


FIG. 5. (a) The relative sizes and the birth points of the (3,3)-cores in the Erdős-Rényi multiplex network are compared with the 3-core in the corresponding ordinary single Erdős-Rényi networks, (b) the relative size of the (2,2)-core of the Erdős-Rényi multiplex network vs the mean degree of its vertices, compared with the 4-core in the corresponding ordinary single Erdős-Rényi networks.

One can obtain the relative size of the corona for the Erdős-Rényi networks using Eq. (9) as

$$n_{k_a, k_b}(k_a, k_b) = \frac{(c_a x_a)^{k_a} e^{-c_a x_a}}{k_a!} \frac{(c_b x_b)^{k_b} e^{-c_b x_b}}{k_b!}. \quad (16)$$

Fig. 4(b) displays the dependence of the corona sizes on the vertex mean degree c . Furthermore, using Eq. (7), we can write the degree distribution of the (k_a, k_b) -core, that is

$$P_{(k_a, k_b)}(q_a, q_b) = \frac{1}{n_{(k_a, k_b)\text{-core}}} \times \frac{(x_a^{q_a} c_a^{q_a} e^{-c_a x_a})(x_b^{q_b} c_b^{q_b} e^{-c_b x_b})}{q_a! q_b!}. \quad (17)$$

Hence, the total vertex mean degree of the (k_a, k_b) -core, for uncorrelated Erdős-Rényi networks with two types of edges a and b , is

$$c_{(k_a, k_b)} = c_a x_a + c_b x_b. \quad (18)$$

As one can notice from Fig. 4(c), in the symmetric case of $c_a = c_b = c$, the total mean degree of the (k_a, k_b) -core changes almost linearly with c .

To round off the study of Erdős-Rényi multiplexes, we compare \mathbf{k} -core percolation on a Erdős-Rényi multiplex network and k -core percolation on its counterpart single network. As one can see in Fig. 5(a), the (3,3)-core

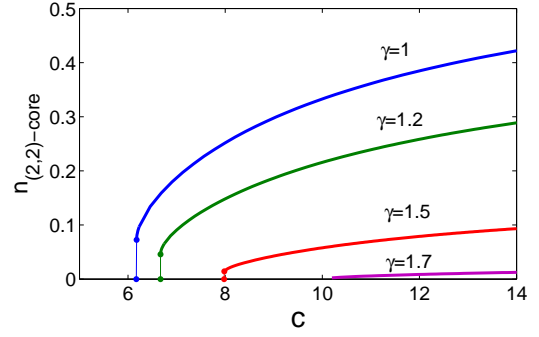


FIG. 6. The relative size of the (2,2)-core (largest viable k -core) for scale-free multiplex networks with an exponential degree cutoff at different values of the exponent γ of the degree distribution. The core disappears as γ increases and approaches 2.

emerges at a much higher mean degree value than the 3-core of the corresponding single Erdős-Rényi graph. In general, (k_a, k_b) -core percolation on the multiplex network has a higher threshold than the k_a or k_b -core on single networks. However, the $(k_a + k_b)$ -core in a single network has a higher threshold than the (k_a, k_b) -core in the corresponding multiplex networks. Fig. 5(b) compares the relative sizes of the (2,2)-core of the Erdős-Rényi multiplex network with the 4-core in the corresponding ordinary single Erdős-Rényi networks.

C. Scale-free networks

Let us consider scale-free multiplex networks with two types of edges. For simplicity we assume that the degrees q_a and q_b are distributed in the same manner with the power-law exponents $\gamma_a = \gamma_b \equiv \gamma$ and the mean degrees $\langle q_a \rangle = \langle q_b \rangle \equiv c$. First we consider the organization of \mathbf{k} -cores for the power-law distributed networks with an exponential degree cutoff, *i.e.*, for instance, $P(q) = \frac{q^{-\gamma} e^{-q/\kappa}}{Li_\gamma(e^{-1/\kappa})}$, where $Li_n(x)$ is the n th polylogarithm of x . For this distribution, the generating function defined by Eq. (3) is $G(x) = \frac{x Li_\gamma(e^{-1/\kappa})}{Li_\gamma(e^{-1/\kappa})}$. The results significantly depend on the low-degree part of the degree distribution. The presence of a cutoff enables us to consider even low values of γ , including $\gamma < 2$. For each value of the exponent γ , we vary the cutoff κ (and thus the mean degree c) as the control parameter. The relative sizes of the \mathbf{k} -cores are obtained by solving Eqs. (4)–(5). Fig. 6 shows $n_{(2,2)\text{-core}}$ for different values of γ . It becomes clear that the size of the core decreases as γ approaches to 2. Hence, for scale-free networks with exponential degree cutoff and, particularly, for the case of pure scale-free networks ($\kappa \rightarrow \infty$), the (k_a, k_b) -core does not exist for $\gamma > 2$.

Next we consider asymptotically scale-free networks

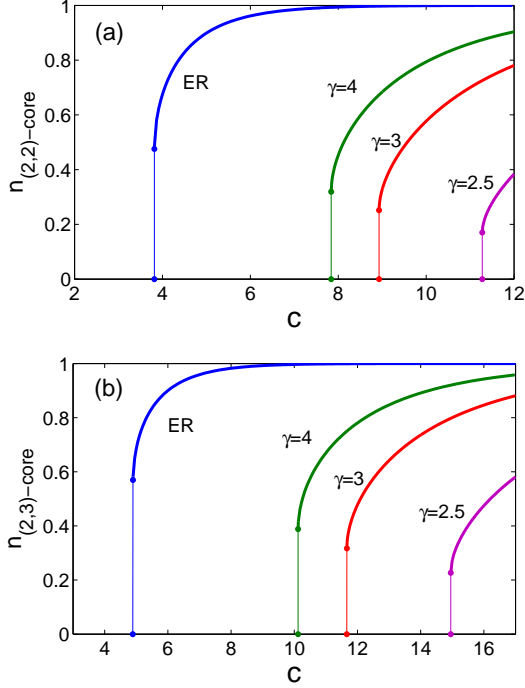


FIG. 7. The relative size of (a) (2,2)–core and (b) (2,3)–core, for ER and asymptotically scale free networks, with different values of γ . For SF networks with smaller γ , the critical point shifted to higher mean degree values.

generated by the static model with

$$P(q) = \left\{ \left[\frac{\langle q \rangle (\gamma - 2)}{2(\gamma - 1)} \right]^{\gamma-1} \Gamma(q - \gamma + 1, \frac{\langle q \rangle (\gamma - 2)}{2(\gamma - 1)}) \right\} / \Gamma(q + 1), \quad (19)$$

where $\Gamma(s)$ is the gamma function and $\Gamma(s, x)$ is the upper incomplete gamma function [32]. This function in the large q limit is asymptotically power law, $P(q) \sim q^{-\gamma}$ for $\gamma > 2$. The generating function is $G(x) = (\gamma - 1)E_n[(1 - x)\frac{\langle q \rangle (\gamma - 2)}{2(\gamma - 1)}]$, where $E_n(x) = \int_1^\infty dy e^{-xy} y^{-n}$ is the exponential integral.

For different values of γ , Fig. 7 shows the relative size of the (2,2)–core and the (2,3)–core and their corresponding emergence points. The value of the jumps at the critical point, increases with increasing γ . Also, when the exponent γ decreases, the transition point moves towards higher values of mean degree. In Fig. 7 we compare the emergence of cores for scale-free and Erdős–Rényi networks. As one can see, the dependency of the cores on c for these networks is similar and, as expected, the curves with larger γ approach to the result for Erdős–Rényi networks. The remarkable difference between Figs. 6 and 7 for different scale-free multiplex networks points out that, similarly to ordinary k -cores, the \mathbf{k} -cores essentially depend on the low-degree parts of the degree distributions.

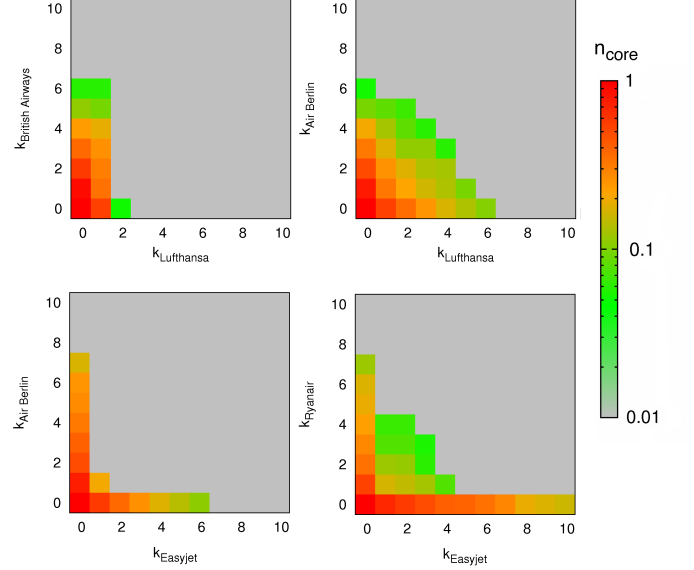


FIG. 8. The relative size of \mathbf{k} -cores of BritishAirway-Lufthansa, AirBerlin-Lufthansa, AirBerlin-EasyJet and Ryanair-EasyJet multiplex transportation networks, for different values of k_a and k_b .

III. REAL MULTIPLEX NETWORKS

As an example of real multiplex networks, we consider air-transportation networks, in which vertices represent airports and edges direct flight connections between airports. Since flights can be operated by different airlines, a detailed representation of this kind of systems is given by multiplexes [33]. For the sake of simplicity we consider air-transportation multiplexes composed of two types of edges (two different airlines). Among the possible choices for composing these multiplexes, we have considered both low-cost airlines and major carriers. The main difference between the organization of these airlines is that low-cost companies diversify their main airports across and their goal is mainly driven by the economic growth. Instead major carriers are typically associated to one country as they are originally designed to serve the national and international mobility of the corresponding citizens. These major airlines are thus designed following the so-called *hub-and-spoke* structure in which one airport (the hub) is surrounded by many low degree vertices forming a kind of star-like graph.

We have considered three different types of multiplexes comprising: (i) two low-cost airlines, (ii) two major airlines, and (iii) one low-cost and one major airline. In particular, we have considered these combinations: *EasyJet – Ryanair* and *EasyJet – AirBerlin* (combination of two low-cost airlines), *Lufthansa – British Airways* (combination of two major airlines) and *Lufthansa – AirBerlin* (combination of low-cost and major airlines).

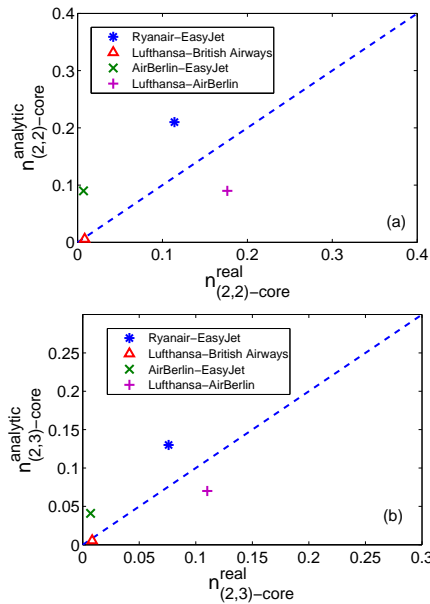


FIG. 9. Comparison between \mathbf{k} -core sizes obtained from empirical data and our theory for the relative sizes of (a) the (2,2)-core and (b) the (2,3)-core of transportation multiplex networks. Theoretical values are calculated for uncorrelated counterparts of the corresponding empirical multiplex networks.

Fig. 8 shows the sizes of \mathbf{k} -core of these networks for different values of k_a and k_b . For the case *EasyJet* – *Ryanair* and *Lufthansa* – *AirBerlin* (which are two airlines operating in the same country), we obtain more central cores, since the combined companies have many common vertices with a large number of connections for each type of edges. On the other hand, for *EasyJet* – *AirBerlin*, the common vertices have a few connections for each type of edges, which are removed in the first steps of \mathbf{k} -core algorithm. Hence, this multiplex network has only a few cores. Similarly, the *Lufthansa* – *BritishAirways* multiplex network has a few cores, since these two major airlines operate from different countries and thus the connectivity of the overlapping vertices is very different.

In Fig. 9, we have compared the size of the (2,2)-core and the (2,3)-core for air-transportation multiplex networks with the corresponding analytical results, obtained from Eqs. (4)–(5) in which we made use of the degree distributions of the empirical multiplex networks. As one can see, in some cases there is a noticeable difference between theory and reality, which arises from clustering, degree-degree correlations, and structural motifs in real-world networks, which we did not take into account. Furthermore, the effects of overlapping between edges from different layers in the real-world multiplex networks can also be significant as noted in [34, 35].

IV. CONCLUSION

In this work we have generalized the theory of \mathbf{k} -core percolation to the multiplex networks. Our aim is to present a pruning algorithm for the \mathbf{k} -core decomposition of multiplex networks that may be useful for describing the topological structure of these networks. To this aim, we introduced a \mathbf{k} -viable component and showed that the \mathbf{k} -core is the giant \mathbf{k} -viable component if $k_i \geq 2$ for each type of edges. In this case, for each two vertices in the \mathbf{k} -viable component there are at least k_i paths, following only edges of type i , joining them.

We have analytically solved the \mathbf{k} -core percolation problem for multiplex networks with arbitrary degree distributions. In the particular case of (1,1)-core, the transition is continuous. This is not in contradiction with previous results for ordinary percolation on multiplex networks, because in this case the (1,1)-core is not a \mathbf{k} -viable component. We showed however that the transitions for higher cores are hybrid. Noticeably, although the (1,2) and (2,1)-cores are not viable, they also display hybrid transitions. Moreover we found that the (k_a, k_b, \dots) -core on uncorrelated multiplex networks has a higher threshold than the k_i -core on the counterpart single networks. Hence, we conclude that multiplex networks are less robust compared to their counterpart single networks, if we analyse the destruction of the \mathbf{k} -cores induced by random removal of vertices.

In summary, the \mathbf{k} -core problem for the multiplex networks turns out to be essentially richer than the k -core problem for single networks. The pruning algorithm allows one to extract the \mathbf{k} -cores in the multiplex networks in an easier way than their viable components. So the \mathbf{k} -core decomposition of these complex networks is algorithmically efficient. In the analytical framework presented here, we focused on uncorrelated and locally tree-like multiplex networks, ignoring the overlap of different types of edges, clustering, and correlations. Our empirical data analysis has revealed that these features may be significant for the sizes and organization of \mathbf{k} -cores. We suggest that our theory could be extended to consider the case of complex multiplex networks with diverse structural correlations.

ACKNOWLEDGMENTS

This work was partially supported by the Portuguese FCT under projects PTDC/MAT/114515/2009, and PEst-C/CTM/LA0025/2011; the Spanish MINECO under projects FIS2011-25167 and FIS2012-38266-C02-01; and the FET IP Project MULTIPLEX (317532). JGG is supported by the Spanish MINECO through the Ramon y Cajal program.

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